

ENDOMORPHISM RINGS OF TORSIONLESS MODULES

BY

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Abstract. Let A be a right order in a semisimple ring Σ , M_A be a finite-dimensional torsionless right A -module and $\bar{M}_A$ be the injective hull of M_A . J. M. Zelmanowitz has shown that $Q = \text{End } \bar{M}_A$ is a semisimple ring and $S = \text{End } M_A$ is a right order in Q . Further, if A is a two-sided order in Σ then S is a two-sided order in Q . We give a conceptual proof of this result. Moreover, we show that if A is a bounded order then so is S . The underlying idea of our proofs is very simple. Rather than attacking $S = \text{End } M_A$ directly, we prove the results for $B = \text{End } (M_A \oplus A_A)$. If $e: M_A \oplus A_A \rightarrow M_A \oplus A_A$ is the projection on M along A_A then, of course, $S \cong eBe$ and it is easy to transfer the required information from B to S . The reason why it is any easier to look at B rather than S is that $M_A \oplus A_A$ is a generator in $\text{mod-}A$ and a Morita type transfer of properties from A to B is available. We construct an Artinian ring resp. Noetherian prime ring containing a right ideal whose endomorphism ring fails to be Artinian resp. Noetherian from either side.

1. Preliminaries. Throughout this paper, all rings are associative and have unity, all modules and bimodules are unitary. We always write homomorphisms on the opposite side of the scalars.

Let A be a ring, U_A be a right A -module, $B = \text{End } U_A$ and $U^* = \text{Hom } (U_A, A)$. Then U is naturally a (B, A) -module and U^* is naturally a (A, B) -module. Let E denote the endomorphism ring of the left A -module ${}_A U^*$. Then we have a homomorphism $\theta: B \rightarrow E$ defined by $(x)(\theta(b)) = xb$ for all $x \in U^*$ and $b \in B$. The map θ is called the *canonical* homomorphism of B into E .

From the module U_A , we can prepare the usual pre-equivalence data (B, A, U, U^*, f, g) . We shall usually suppress mention of f and g while working with this pre-equivalence data. For details, see [2, p. 67].

As pointed out in the abstract, we shall mainly deal with the case when U_A is a generator in $\text{mod-}A$ with various further conditions on A and U_A and we shall examine the behavior of B . We shall be particularly interested in the case when U_A is torsionless. Recall that a module U_A is said to be *torsionless* if

$$(0) = \bigcap \{\ker f \mid f \in U^*\}.$$

Also recall that if U_A is a generator in $\text{mod-}A$ then ${}_A U^*$ is a generator in $A\text{-mod}$ and ${}_B U$ is a finitely generated projective (cf. [2, p. 63]).

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The usual presentation of Morita theorems, i.e. [2], develops a close connection between right A -submodules of U_A and right ideals of B when U_A is a progenerator in $\text{mod-}A$. Recently, C. Faith has proved that this connection can be retained in suitably modified form when U_A is just a generator in $\text{mod-}A$. Since we shall repeatedly need it, we quote it without proof; for proof see [3].

1.1. THEOREM (C. FAITH). *Let U_A be a generator in $\text{mod-}A$ and $B = \text{End } U_A$. Given a right A -submodule W of U , put $\rho_0(W) = \{b \in B \mid bU \subseteq W\}$ and $\rho(W) = \rho_0(W)T$ where T is the trace ideal of ${}_B U$. Then $\rho(W)$ is a right ideal of B , $\rho(W)T = \rho(W)$ and $\rho(W)U = W$. The map $W \mapsto \rho(W)$ defines a lattice isomorphism*

$$\rho: \{\text{right } A\text{-submodules of } U\} \rightarrow \{K \mid K \text{ a right ideal of } B \text{ with } KT = K\}.$$

The inverse of ρ associates a right ideal K of B with KU . The restriction of ρ yields a lattice isomorphism

$$\rho: \{(B, A)\text{-submodules of } U\} \rightarrow \{I \mid I \text{ an ideal of } B \text{ with } IT = I\}.$$

Given an ideal J of A , put $\psi(J) = UJ$. Then ψ defines a lattice isomorphism

$$\psi: \{\text{ideals of } A\} \rightarrow \{W \mid W \text{ a } (B, A)\text{-submodule of } U \text{ with } TW = W\}.$$

The map $S = \rho \circ \psi$ defines a multiplicative semigroup isomorphism of ideals of A with $\{I \mid I \text{ an ideal of } B \text{ with } TIT = I\}$.

The following corollary is immediate.

1.2. COROLLARY (C. FAITH). *Let U_A be a generator in $\text{mod-}A$. Then*

(1) *A is semiprime if and only if $TX = (0)$ for every nilpotent ideal X of B . In particular, if B is semiprime then so also is A .*

(2) *A is prime if and only if the following holds in B : If J, K are ideals of B with $JK = (0)$ then $TJ = (0)$ or $TK = (0)$. In particular, if B is prime then so also is A .*

We now impose some additional hypotheses on U_A and examine the behavior of B . If M_A is a A -module, $d(M_A)$ denotes the uniform dimension of M_A .

1.3. THEOREM. *Let U_A be a generator in $\text{mod-}A$. Then the following assertions hold:*

(1) *U_A is torsionless $\Leftrightarrow \theta$ is a monomorphism $\Leftrightarrow U_B^*$ is faithful. If these equivalent conditions hold then A is prime or semiprime if and only if B is respectively so.*

(2) *U_A is finite dimensional if and only if B is a right finite-dimensional ring. In this case, $d(U_A) = d(B_B)$.*

(3) *A right ideal K of B is essential in B if and only if KU is an essential right A -submodule of U .*

(4) *If U_A is nonsingular then B is a right nonsingular ring.*

Proof. (1) Suppose U_A is torsionless. Let $b \in B$ with $\theta(b) = 0$. Then $U^*b = (0)$ so $U^*(bU) = 0$ so $bU = (0)$ which means $b = 0$. If θ is a monomorphism then clearly U_B^* is faithful. Suppose U_B^* is faithful. Put $W = \bigcap \{\ker f \mid f \in U^*\}$. Evidently, W is

a (B, A) -submodule of U so, by Theorem 1.1, $W=IU$ for an ideal I of B . Now, $(0)=U^*W=U^*(IU)=(U^*I)U$ yields $U^*I=(0)$ so $I=0$ and $W=IU=(0)$. Thus U_A is torsionless.

Assume that U_A is torsionless and that A is prime. Let J, K be ideals of B such that $JK=(0)$. By Corollary 1.2, $TJ=(0)$ or $TK=(0)$ where T is the trace ideal of ${}_B U$. By Morita theorems, U_B^* is a finitely generated projective B -module and $U \cong_{\text{canon}} \text{Hom}(U_B^*, B)$. It follows that T is the trace ideal of U_B^* and $U^*T=U^*$. Now $TJ=(0)$ yields $0=U^*TJ=U^*J$ and faithfulness of U_B^* yields $J=(0)$. Similarly, $TK=(0)$ yields $K=(0)$. Hence B is prime. The converse is contained in the Corollary 1.2. The other assertion is similarly proved.

(2) Let J, K be arbitrary right ideals of B . Then $JU \cap KU = JTU \cap KTU = [JT \cap KT]TU$ by the lattice isomorphism in Theorem 1.1. Since $[JT \cap KT]TU \subseteq (J \cap K)U \subseteq JU \cap KU$, it follows that $JU \cap KU = (J \cap K)U$.

Now suppose that U_A is a finite-dimensional module, say $d(U_A)=n$. If possible, let $\bigoplus_{i=1}^{n+1} K_i$ be a direct sum of $n+1$ nonzero right ideals of B . If $x \in K_i U \cap \sum_{j \neq i} K_j U$ then, using the above observation and the lattice isomorphism in Theorem 1.1, $x \in (K_i \cap \sum_{j \neq i} K_j)U = (0)$. Thus $\bigoplus_{i=1}^{n+1} K_i U$ is a direct sum of nonzero A -submodules of U_A , contrary to $d(U_A)=n$. Thus B is a right finite-dimensional ring and $d(B_B) \leq d(U_A)$.

Suppose that B_B is finite dimensional, say $d(B_B)=m$. If possible, let $\bigoplus_{i=1}^{m+1} W_i$ be a direct sum of nonzero A -submodules of U_A . By Theorem 1.1, $\rho(W_i)U = W_i$, so $\rho(W_i)$ are nonzero right ideals of B . If $b \in \rho(W_i) \cap \sum_{j \neq i} \rho(W_j)$ then $bU \subseteq W_i \cap \sum_{j \neq i} W_j = (0)$ so $b=0$. Thus, $\sum_{i=1}^{m+1} \rho(W_i)$ is direct contrary to $d(B_B)=m$. Hence U_A is finite dimensional and $d(U_A) \leq d(B_B)$. Combined with the reverse inequality proved above, we get $d(U_A)=d(B_B)$.

(3) Let K be an essential right ideal of B . If N_A is a right A -submodule of U_A such that $KU \cap N = (0)$ then $\rho(N)U = N$ and the observation in the proof of (2) shows that $(0) = [K \cap \rho(N)]U$. Thus $K \cap \rho(N) = (0)$ so $\rho(N) = (0)$ and consequently, $N = (0)$; i.e., KU is essential in U .

Conversely, let K be a right ideal of B such that KU is essential in U_A . If J is a right ideal of B with $J \cap K = (0)$ then $(0) = [J \cap K]U = JU \cap KU$ so $JU = (0)$ which yield $J = (0)$. Hence K is essential in B .

(4) Let $b \in B$ and K be an essential right ideal of B with $bK = (0)$. Then, as shown above, $N = KU$ is an essential A -submodule of U . Let x be an arbitrary element of U and let $(x:N) = \{a \in A \mid xa \in N\}$. The essentiality of N in U_A shows that $(x:N)$ is an essential right ideal of A . Since $b[x(x:N)] \subseteq bN = bKU = (0)$ we have $b(x) \cdot (x:N) = (0)$. Since U_A is nonsingular, we have $b(x) = 0$, so $b = 0$. This completes the proof of the theorem.

2. A Theorem of Zelmanowitz. Let A be a right order in a semisimple ring Σ and M_A be a right A -module. An element $x \in M$ is called a *torsion element* if $xc=0$ for some regular element c in A . M is a *torsion module* if every element of

M is a torsion element. M is *torsion-free* if no nonzero element of M is a torsion element.

We begin with some trivialities concerning modules over A .

2.1. PROPOSITION. *Let A be a right order in a semisimple ring. If N is an essential submodule of a right A -module M_A then M/N is a torsion module.*

Proof. Let $x \in M$. Since N is essential in M therefore $(x:N)$ is an essential right ideal of A , so $(x:N)$ contains a regular element, say c , of A . Then $xc \in N$ i.e. $(x+N)c = (0)$ in M/N .

2.2. COROLLARY. *If $h: M_A \rightarrow L_A$ is a homomorphism, L is torsion-free and $\ker h$ is essential in M_A then $\ker h = M$ and $h = 0$.*

Proof. $M/\ker h$ is torsion and sits in L .

2.3. COROLLARY. *If N_A is an essential submodule of a module M_A then ${}_A M^* \xrightarrow{\text{canon}} {}_A N^*$ is a monomorphism.*

2.4. PROPOSITION. *Let A be a right order in a semisimple ring and M_A be a right A -module. M_A is finite dimensional and torsionless if and only if M_A is isomorphic with a submodule of a finitely generated free right A -module.*

Proof. Let M be finite dimensional and torsionless. Let $\mathcal{F}$ be the set of all homomorphisms $M \rightarrow A^{(n)}$ for all nonnegative integers n . Choose $h: M \rightarrow A^{(m)}$ such that $d(\ker h) = \min \{d(\ker f) \mid f \in \mathcal{F}\}$. Put $K = \ker h$. If $K \neq 0$, choose a uniform $V_A \subseteq K$ and an element $0 \neq x \in V$. Since M is torsionless, we can choose a homomorphism $g: M \rightarrow A$ such that $g(x) \neq 0$. Corollary 2.2 shows that g is injective on V . Define $h': M \rightarrow A^{(m+1)}$ by $h'(y) = (h(y), g(y))$ for all $y \in M$. Since $\ker h' = K \cap \ker g$, it follows that $\ker h' \cap V = (0)$ so $d(\ker h') < d(K)$, contrary to our choice. Thus $h: M \rightarrow A^{(m)}$ is a monomorphism. The converse is trivial.

We now prove the following crucial lemma.

2.5. LEMMA. *Let A be a right order in a semisimple ring Σ , U_A be a finite-dimensional torsionless right A -module which is also a generator in $\text{mod-}A$, $\hat{U}_A$ be the injective hull of U_A . Put $B = \text{End } U_A$ and $\Lambda = \text{End } \hat{U}_A$. Then*

- (1) Λ is a semisimple ring and B is a right order in Λ .
- (2) If Σ is simple then so is Λ .
- (3) If A is a two-sided order in Σ then B is a two-sided order in Λ .

Proof. (1) Since every essential right ideal of A contains a regular element, therefore U_A is a nonsingular module. By Theorem 1.3, B is a right finite-dimensional, right nonsingular semiprime ring. By Goldie's theorem, B is a right order in some semisimple ring. Thus every essential right ideal of B contains a regular element of B .

It is clear that every homomorphism $b: U_A \rightarrow U_A$ can be extended to a homomorphism $\hat{b}: \hat{U}_A \rightarrow \hat{U}_A$. It is immediate from Corollary 2.2 that $\hat{b}$ is uniquely determined by b and that $b \mapsto \hat{b}$ defines a monomorphism $\sigma: B \rightarrow \Lambda$. We shall

identify B with $\sigma(B)$ by σ . This identification is tacitly made in the statement of the lemma.

It is well known that $\hat{U}_A$ is, in fact, the module of quotients of U_A , $\hat{U}_A$ can be naturally considered as a right Σ -module, $d(U_A) = d(\hat{U}_A) = d(\hat{U}_\Sigma)$ and that the Σ -endomorphisms of $\hat{U}$ are precisely the A -endomorphisms of $\hat{U}$ (cf. [7]). Hence Λ is a semisimple ring and Λ is simple if Σ is simple.

We proceed to show that B is a right order in Λ . Let b_0 be a regular element of B . If $\ker b_0 \neq (0)$ then $\ker b_0 \cap U \neq (0)$. By Theorem 1.1, we have a nonzero right ideal I of B with $IU = \ker b_0 \cap U$. So, $b_0IU = (0)$ which yields $b_0I = (0)$, a contradiction. It follows that b_0 is a unit in Λ . Let $n = d(U_A)$. Choose cyclic uniform A -submodules x_iA of U , $1 \leq i \leq n$, such that $\sum_{i=1}^n x_iA$ is a direct sum. Let λ be an arbitrary element of Λ . Since $\hat{U}$ is the module of quotients of U , using the common denominator property in A , we can choose a regular element c in A such that $\lambda(x_ic) \in U$ for $1 \leq i \leq n$. Since U_A is torsion free, $x_ic \neq 0$ for all i . Thus $N = \sum_{i=1}^n x_icA$ is a direct sum of n uniform submodules of U_A . Therefore N is an essential A -submodule of U_A and $\lambda(N) \subseteq U$. By Theorem 1.3, $\{b \in B \mid bU \subseteq N\}$ is an essential right ideal of B , so it contains a regular element of B , say b_0 . Evidently, $\lambda b_0(U) \subseteq U$ so $\lambda b_0 \in B$. Hence B is a right order in Λ . This proves (1) and (2).

Now assume that A is a two-sided order in Σ . Since U_A is finite dimensional, we can choose a finitely generated essential A -submodule W_A of U_A . By Corollary 2.3, ${}_AU^* \xrightarrow{\text{canon}} {}_AW^*$ is a monomorphism of left A -modules. Since W_A is finitely generated, we have an epimorphism $A_A^{(k)} \rightarrow W_A$ which yields a monomorphism ${}_AW^* \rightarrow {}_AA^{(k)}$. It follows that ${}_AU^*$ is a finite-dimensional left A -module. Let $d = d({}_AU^*)$. If possible, let $\bigoplus_{i=1}^{d+1} K_i$ be a direct sum of left ideals of B . Since U_A is torsionless, U_B^* is faithful and so $U^*K_i \neq (0)$ for all i . If $x \in U^*K_i \cap \sum_{j \neq i} U^*K_j$, then $Ux \subseteq UU^*K_i \cap U(\sum_{j \neq i} U^*K_j) \subseteq TK_i \cap \sum_{j \neq i} TK_j = (0)$ where T is the trace ideal of ${}_BU$. By Morita theorems, ${}_AU_B^* \xrightarrow{\text{canon}} \text{Hom}({}_BU, B)$. So, $Ux = (0)$ yields $x = 0$, i.e. $\sum_{i=1}^{d+1} U^*K_i$ is a direct sum of nonzero left A -submodules of ${}_AU^*$, contrary to the definition of d . Hence B is finite dimensional on the left. Since B is already a right order in Λ , it is semiprime and has the ascending chain condition on left annihilators. By Goldie's theorem, B is a left order in some semisimple ring. It is then trivial to see that B must be a two-sided order in Λ . This completes the proof of the lemma.

At this point, it is convenient to recall the following fact.

2.6. LEMMA. *Let R be a right order in a semisimple ring Q and e be an idempotent in R . Then eRe is a right order in the semisimple ring eQe .*

Proof. See Lemma 1 in [4] and Lemmas 3 and 4 in [8].

We are now in a position to prove the following theorem of J. Zelmanowitz [9].

2.7. THEOREM. *Let A be a right order in a semisimple ring Σ , M_A be a finite-dimensional torsionless right A -module and $\hat{M}_A$ be the injective hull of M_A . Put $S = \text{End } M_A$ and $Q = \text{End } \hat{M}_A$. Then*

- (1) Q is a semisimple ring and S is a right order in Q .
- (2) If Σ is simple then so is Q .
- (3) If A is a two-sided order in Σ then S is a two-sided order in Q .

Proof. Let $U_A = M_A \oplus A_A$, $B = \text{End } U_A$ and $e: U_A \rightarrow U_A$ be the projection on M_A along A_A . Then e is an idempotent in B and $S \stackrel{\text{canon}}{\cong} eBe$. Let $\hat{A}_A$ be the injective hull of A_A . Then $\hat{U}_A = \hat{M}_A \oplus \hat{A}_A$ is the injective hull of U_A . If $\Lambda = \text{End } \hat{U}_A$ then $Q \stackrel{\text{canon}}{\cong} e\Lambda e$. It is now easy to complete the proof using Lemmas 2.5 and 2.6.

3. Bounded orders. A right order A in a semisimple ring is said to be *right bounded* if every essential right ideal of A contains a two-sided ideal of A which is essential as a right ideal. A left bounded left order in a semisimple ring is similarly defined. A two-sided order A in a semisimple ring is said to be *bounded* if it is right bounded and left bounded. It is easy to see that if A is a prime ring which is an algebra over a commutative Noetherian ring R and if A_R is finitely generated as a module then A is a bounded order in a simple Artinian ring. See also [5]. For some extremely one-sided examples, see [6].

Let A be a right order in a semisimple ring Σ . A module M_A is called a *totally torsion module* if $\text{ann } M_A$ is an essential right ideal of A , equivalently, $Mc = (0)$ for some regular element c in A . Note that, if Σ is simple then M_A is totally torsion if and only if it is unfaithful.

3.1. LEMMA. *Let A be a right order in a semisimple ring. Then the following conditions on A are equivalent:*

- (1) A is right bounded.
- (2) Every cyclic torsion right A -module is totally torsion.
- (3) Every finitely generated torsion right A -module is totally torsion.
- (4) If N_A is an essential submodule of a finite-dimensional torsionless right A -module M_A then M/N is totally torsion.

Proof. We give a cyclic proof.

(1) $\Rightarrow$ (2). Let I be a right ideal of A such that A/I is a torsion module. Then there exists a regular element c in A such that $(1+I)c = 0+I$, i.e. $c \in I$. Thus I is essential in A . By hypothesis, we have a two-sided ideal T of A such that T is essential as a right ideal of A and $T \subseteq I$; so, $\text{ann } (A/I) \supseteq T$.

(2) $\Rightarrow$ (3). Let $K = \sum_{i=1}^n x_i A$ be a torsion module. Choose two-sided ideals T_i of A such that each T_i is an essential right ideal of A and $(x_i A)T_i = (0)$. Let $T = T_1 T_2 \cdots T_n$. Since each T_i contains a regular element of A , T must contain a regular element of A . Evidently $KT = (0)$ so K is totally torsion.

(3) $\Rightarrow$ (4). By Proposition 2.4, we may assume that $M_A \subseteq A^{(n)}$ for some positive integer n . Choose a submodule L of $A^{(n)}$ such that $M \cap L = (0)$ and $M+L$ is essential in $A^{(n)}$. Then $N+L$ is also essential in $A^{(n)}$. By Proposition 2.1, $A^{(n)}/(N+L)$ is a finitely generated torsion module, so totally torsion. Since $M/N \cong (M+L)/(N+L) \hookrightarrow A^{(n)}/(N+L)$, it follows that M/N is totally torsion.

(4) $\Rightarrow$ (1). Evident. This completes the proof.

3.2. LEMMA. *Let A be a right order in a semisimple ring Σ , U_A be a finite-dimensional torsionless right A -module which is also a generator in $\text{mod-}A$ and $B = \text{End } U_A$. Then B is a right order in a semisimple ring Λ . If A is right bounded then so is B . If A is a bounded two-sided order in Σ then B is a bounded two-sided order in Λ .*

Proof. We have already seen that B is a right order in a semisimple ring Λ and that B is a two-sided order in Λ if A is a two-sided order in Σ .

Assume that A is right bounded right order in Σ . We proceed to show that B is right bounded. Let K be an essential right ideal of B . By Theorem 1.3, KU is an essential right A -submodule of U_A . By Lemma 3.1, U/KU is totally torsion. Thus, there exists a two-sided ideal I of A such that I contains a regular element of A and $UI \subseteq KU$. Let T be the trace ideal of ${}_B U$. By Theorem 1.1, there exists an ideal J of B such that $TJT = J$ and $JU = UI$. Now $JU \subseteq KU$ yields $J \subseteq KT \subseteq K$. We shall show that J is an essential right ideal of B . Let L be a right ideal of B such that $J \cap L = (0)$; so, $LJ = (0)$ which gives $(TBLT)(TJT) = (0)$. Now, the map $S = \psi \circ \rho$ in Theorem 1.1 is a semigroup isomorphism, I is not annihilated by any nonzero ideal of A and $J = S(I)$. It follows that $TBLT = (0)$; so, $(0) = TBLTU = TLU$; so, $TL = (0)$. However, by Morita theorems, ${}_B U \stackrel{\text{canon}}{\cong} \text{Hom}(U_B^*, B)$ so T is also the trace ideal of U_B^* and $U^*T = U^*$. Thus, $TL = (0)$ yields $U^*L = (0)$. By Theorem 1.3, U_B^* is faithful so that $L = (0)$. We have thus shown that J is an essential right ideal of B . Hence B is right bounded.

Now assume that A is a bounded two-sided order in Σ . We continue to denote the trace ideal of ${}_B U$ by T . If $T \cap F = (0)$ for some left ideal F of B then $TF = 0$ so $U^*TF = U^*F = (0)$ which yields $F = (0)$. Thus T is an essential left ideal of B and so contains a regular element of B since B is now a two-sided order in Λ .

Let H be an essential left ideal of B . Since T and H contain regular elements of B , it follows that TH is an essential left ideal of B . Let v be a nonzero element of U^* . Since U_B^* is a finitely generated projective and ${}_B U_A \stackrel{\text{canon}}{\cong} \text{Hom}(U_B^*, B)$, therefore there exists an element $u \in U$ such that $0 \neq uv = b \in B$. Since TH is an essential left ideal of B , we may assume that $0 \neq uv = b \in TH$. Choose $w \in U^*$ such that $wb \neq 0$; this is possible since U_B^* is faithful. Put $a = wu \in U^*U = A$. Then $0 \neq av = wuv = wb \in U^*H \cap Av$. Thus U^*H is an essential left A -submodule of ${}_A U^*$. Now ${}_A U^*$ is always torsionless and, as in the proof of Lemma 2.5, ${}_A U^*$ is finite dimensional. By Lemma 3.1, there exists a two-sided ideal X of A such that X contains a regular element of A and $XU^* \subseteq U^*H$. So, $H \supseteq TH = UU^*H \supseteq UXU^* = S(X)UU^* = S(X)T = S(X)$, where $S(X)$ is the unique ideal of B given by the semigroup isomorphism $S = \psi \circ \rho$ in Theorem 1.1. As above, $S(X)$ contains a regular element of B . Hence B is bounded. This completes the proof.

3.3. LEMMA. *Let A be a right bounded right order (resp. bounded two-sided order) in a semisimple ring and e be an idempotent in A . Then eAe is a right bounded right order (resp. bounded two-sided order) in a semisimple ring.*

Proof. Let A be a right bounded right order in a semisimple ring. Lemma 2.6 shows that eAe is a right order in a semisimple ring. Let K be an essential right ideal of eAe . Then K contains a regular element of eAe , say ece . It is then clear that $ece + (1-e)$ is a regular element of A and it belongs to the right ideal $KA + (1-e)A$. Thus $KA + (1-e)A$ is an essential right ideal of A . We have an ideal I of A such that I contains a regular element of A and $I \subseteq KA + (1-e)A$. Then $eIe \subseteq eKAe = K$. If $X = eXe$ is an ideal of eAe such that $(eIe)(eXe) = (0)$ then

$$[I(AeXeA)]^2 = IeX(eIe)(eXe)A = (0).$$

Since A is semiprime, $I(AeXeA) = (0)$ and since I contains a regular element of A , $eXe = (0)$. It follows that eIe is an essential right ideal of eAe . Thus eAe is right bounded. The two-sided case is now immediate. This completes the proof.

3.4. THEOREM. *Let A be a right bounded right order in a semisimple ring Σ , M_A be a finite-dimensional torsionless right A -module and $B = \text{End } M_A$. Then B is a right bounded right order in a semisimple ring Λ . If A is a bounded two-sided order in Σ then B is a bounded two-sided order in Λ .*

Proof. Consider $U_A = M_A \oplus A_A$. The theorem is immediate from Lemmas 3.2 and 3.3.

4. An example. We give an example to show that the endomorphism ring of a finitely generated torsion-free module over a Noetherian ring may fail to be Noetherian.

Let k be a field, $K = k(t_{ij} : i, j \in \mathbb{Z})$ be the rational function field in commuting indeterminates t_{ij} , $i, j \in \mathbb{Z}$. Let ρ, σ be the unique k -automorphisms of K defined by $\rho(t_{ij}) = t_{i+1, j}$ and $\sigma(t_{ij}) = t_{i, j+1}$ for all i, j . Observe that $\sigma\rho = \rho\sigma$.

Let $D = K[x, \rho; y, \sigma]$ be the polynomial ring in indeterminates x, y with coefficients in K ; the multiplication in D is defined by the relations $xy = yx$, $\alpha x = x\rho(\alpha)$ and $\alpha y = y\sigma(\alpha)$ for all $\alpha \in K$. These relations are consistent since $\sigma\rho = \rho\sigma$. As in the Hilbert basis theorem, D is a left and right Noetherian domain. Let F be the quotient skew field of D . Let $I = x^2D + xyD + y^2D$ and let $N = (x+y)D + I$. Evidently, I is a two-sided ideal of D contained in the right ideal N of D .

Let

$$B = k + xD + yD, \quad A = \begin{pmatrix} D & D \\ I & D \end{pmatrix}, \quad Q = \begin{pmatrix} F & F \\ F & F \end{pmatrix},$$

$$M = \begin{pmatrix} N & D \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x+y & 1 \\ 0 & 0 \end{pmatrix} A, \quad \text{and} \quad X = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}.$$

It is easily seen that A is a left and right Noetherian prime ring which is an order in the simple Artinian ring Q and M_A is a uniform right ideal of A . Since X_Q is a simple module, X_A is a torsion-free uniform injective A -module and thus an injective hull of M_A (cf. [7]). Using Corollary 2.2, every A -homomorphism $b: M_A \rightarrow M_A$ can be uniquely extended to a Q -homomorphism $\hat{b}: X_Q \rightarrow X_Q$. The map $b \mapsto \hat{b}$ is easily seen to be a monomorphism

$$\text{End } M_A \rightarrow \text{End } X_Q \cong \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus

$$\text{End } M_A \cong \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \mid \alpha \in F, \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} M \subseteq M \right\} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \cong B.$$

It is readily seen that the ideal $xD + yD$ of B is not finitely generated from either side. Thus $\text{End } M_A$ is not Noetherian from either side although it is a two-sided order in the skew field F .

We proceed to show how the above example can be modified to obtain an instructive Artinian ring.

Let $J = x^3D + x^2yD + xy^2D + y^3D$. Clearly, J is a two-sided ideal of D as well as of B . Put $\bar{D} = D/J$, $\bar{B} = B/J$, $\bar{N} = N/J$, $\bar{I} = I/J$,

$$R = \begin{pmatrix} \bar{D} & \bar{D} \\ \bar{I} & \bar{D} \end{pmatrix} \quad \text{and} \quad \bar{M} = \begin{pmatrix} \bar{N} & \bar{D} \\ 0 & 0 \end{pmatrix}.$$

Then R is a left and right Artinian ring and

$$\bar{M} = \begin{pmatrix} \bar{x} + \bar{y} & \bar{I} \\ 0 & 0 \end{pmatrix} R$$

is a right ideal of R . Identify $\bar{B}$ with

$$\left\{ \begin{pmatrix} \bar{b} & 0 \\ 0 & 0 \end{pmatrix} \mid \bar{b} \in \bar{B} \right\}$$

so that $\bar{B}$ is a (not necessarily unitary) subring of R . It is readily seen that the map $\bar{b}_1: \bar{M}_R \rightarrow \bar{M}_R$ defined by $\bar{b}_1(m) = \bar{b}m \quad \forall m \in \bar{M}$ is a R -homomorphism and that $\bar{b} \mapsto \bar{b}_1$ defines a monomorphism $\bar{B} \rightarrow \text{End } \bar{M}_R$. Let $f \in \text{End } \bar{M}_R$; say,

$$f \begin{pmatrix} \bar{x} + \bar{y} & \bar{I} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} r_1 & r_2 \\ 0 & 0 \end{pmatrix} \quad \text{with } r_1 \in \bar{N}, r_2 \in \bar{D}.$$

Using

$$\left\{ f \begin{pmatrix} \bar{x} + \bar{y} & \bar{I} \\ 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} 0 & \bar{I} \\ 0 & 0 \end{pmatrix} = \left\{ f \begin{pmatrix} 0 & \bar{I} \\ 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} 0 & 0 \\ 0 & \bar{x} + \bar{y} \end{pmatrix},$$

it follows that $r_1 = r_2(\bar{x} + \bar{y})$ so that $r_2 \in \bar{B}$. Then f is induced by $\begin{pmatrix} r_2 & 0 \\ 0 & 0 \end{pmatrix}$; so, $\text{End } \bar{M}_R \cong \bar{B}$. It is clear that the ideal $\bar{I}$ of $\bar{B}$ is not finitely generated from either side. Thus $\text{End } \bar{M}_R$ is not Artinian from either side although R is Artinian from both sides.

Some concluding remarks are in order.

1. Apparently, J. E. Björk has an example of a two-sided Artinian ring R with a R -module M_R such that $\text{End } M_R$ is not Artinian from either side. Although I have not seen his example, I was aware of its existence while constructing the above example. Our Noetherian example appears to be new.

2. J. M. Zelmanowitz [9] contains some examples to show that endomorphism rings of finitely generated torsion-free uniform modules over right Ore domains may be unpleasant.

3. Recently, S. A. Amitsur [1] has shown how "Morita context" can be utilized to get the theorem of Zelmanowitz (among other things). Our proofs differ from Amitsur's both in the underlying idea as well as its execution.

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